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ON TWO CONTROVERSIAL ESTIMATES IN NUMBER THEORY

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Abstract

An unsubstantiated 1958 claim of Vinogradov [15] and Korobov [8] concerning the remainder term in the prime number theorem, never clearly withdrawn by its authors, has been widely reproduced in the literature since then and still occasionally appears even in serious publications.

Korobov's paper [8] is also linked with another less familiar controversy regarding two asymptotic estimates of a sum involving the Euler ϕ-function, due respectively to Walfisz [19] and Saltykov [13].

My two purposes here are to offer some comments on the first issue, based on bibliographic references, and to settle the second one by pointing out a mistake in [13]. This note is largely based on a talk I gave a few years ago at seminars and at a conference.

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1. The Average Value of the Euler Function

Let $\phi(n)$ denote the number of positive integers not exceeding n and prime to n. It is well known that its average value is $3/\pi^2$, in the sense that

$$R(x) := \sum_{n\le x}\phi(n) - \frac{3}{\pi^2}x^2 = o(x^2).$$

If a precise estimate for $R(x)$ is required, it is more convenient to work with the related function

$$H(x) := \sum_{n\le x}\frac{\phi(n)}{n} - \frac{6}{\pi^2}x,$$

and to appeal to the Pillai-Chowla relation $R(x) = xH(x) + o(x)$ [11]. The remainder term H is small. For instance, on noting that $\phi(n)/n = \sum_{d|n}\mu(d)/d$, where μ is the Moebius function, and setting $\psi(x) := x - [x] - 1/2$, a simple argument–simple but appealing to an equivalent form of the prime number theorem–shows that

$$H(x) = -\sum_{n\le x}\frac{\mu(n)}{n}\psi\left(\frac{x}{n}\right) + o(1) = O(\log x).$$

One can improve on the trivial estimate of the last equality by using the Fourier series representing (almost everywhere) the function ψ and techniques for estimating exponential sums. In 1953, Walfisz obtains the first such improvement. In 1958 [18], in an article in which, he also addresses other classical problems in number theory (among which the evaluation of $\zeta(1+it)$ and that of the remainder term in the prime number theorem) he shows that

$$H(x) = O((\log x)^{3/4}(\log\log x)^{3/2}). \tag{1}$$

Finally, in his book [19], published posthumously in 1963, he once again considers this problem, exploits (after having reproved it under a form more convenient to his purpose) a general estimate on Weyl sums $\sum_y e(f(y))$ (where $e(x) := \exp(2i\pi x)$) – see (5) below–recently established by Korobov [8] with Vinogradov's method, and proves that

$$H(x) = O\big((\log x)^{2/3}(\log\log x)^{4/3}\big). \tag{2}$$

In the meantime, Saltykov [13] directly using a result from the same paper [8], and following rather closely the lines in Walfisz' proof of (1), but with some new ideas simplifying the argument (in particular Walfisz' long technical proof of Hilfssatz 4.4.7 in [19] is essentially avoided), publishes in 1960 the proof of a result, which he asserts yields

$$H(x) = O\big((\log x)^{2/3}(\log\log x)^{1+\epsilon}\big), \tag{S}$$

as a corollary.

2. The Error Term in the Prime Number Theorem

The theorem of Korobov which Saltykov uses happens to be published in a very controversial paper.

In 1958, Vinogradov [15] and Korobov [8] each make the same claim on the zeros of the Riemann zeta function, which directly implies an improved unconditional estimate for the remainder term in the prime number theorem,

$$\pi(x) - \ell i(x) = O\big(x\exp(-c\log^{\frac{3}{5}} x)\big). \tag{VK}$$

(Here $\pi(x)$ denotes the prime counting function up to x and

$$\ell i(x) := \lim_{\delta\to\infty}\left(\int_0^{1-\delta}\frac{dt}{\log t} + \int_{1+\delta}^{x}\frac{dt}{\log t}\right) \sim \frac{x}{\log x}\,(x \to \infty).)$$

The estimate (VK) can be obtained by a classical method (see, for instance, [7, Subsection 12.3]), if one knows that there is no zero $s = \sigma + it$ of the Riemann zeta function $\zeta(s)$ in the region of the complex plane described by

$$\sigma \geq 1 - A \log^{-\frac{2}{3}}|t| \tag{VK1}$$

(A is some constant): This is what both Vinogradov [15] and Korobov [8] assert without proof. Walfisz and Richert correspond on the subject, but do not succeed in reconstructing the proof. In fact, nobody succeeds. Finally, Walfisz offers a few lines in the notes at the end of his book ([19], p. 226-227) summarizing an argument communicated to him by Richert, which establishes the absence of zeros of $\zeta(s)$ in a region smaller than (VK1),

$$\sigma \geq 1 - A(\log|t|)^{-\frac{2}{3}}(\log\log|t|)^{-\frac{1}{3}}. \tag{3}$$

A complete proof of this can be given, as Heath-Brown mentions in [14, p. 135], by using Richert's 1967 estimate [12] on $\zeta(\sigma + it)$ $(0 \leq \sigma \leq 2)$ and Theorems 3.10 and 3.11 of [14]. Thus (3) describes the largest region in which we presently know that there is no zero of $\zeta(s)$. This in turn yields

$$\pi(x) - \ell i(x) = O\left(x \exp\left(-c(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}\right)\right), \tag{4}$$

which is, as of today, the best proved unconditional estimate for the remainder term in the prime number theorem.

In 1964, Ingham becomes impatient and, referring to (VK1) in the reviews of the A.M.S. [6], considers that "*it is highly desirable that the claim to the stronger and neater result should be substantiated or withdrawn without further delay*". But Ingham's request remains unanswered.

Korobov does not address the issue again, which is for instance totally ignored in his 1992 book on exponential sums [9].

Vinogradov still claims in 1975 [16] that he can establish (VK). In 1978, this claim appears once more in print (in an article by Lavrik) in the “Encyclopedia of Mathematics”, of which Vinogradov is the editor-in-chief. In 1993, when the annotated english translation of the “Encyclopedia” is published, a few lines’ editorial comment following Lavrik’s article [4, p. 527-531] at last confirms that the largest known zero-free region for $\zeta(s)$ is indeed (3). Unfortunately, the comment compounds the confusion in this matter, for it does not also specify that only (4) is known, nor does it deny Lavrik’s assertion that (VK) is proved. In the introduction of [17] (1980, translated into english in 1985), Vinogradov finally mentions (4) rather than (VK) as a proved result. However, he makes no reference to Korobov’s or Richert’s work, simply asserting that “[…] my method gives [(4)]”.

As a consequence, there has been a number of publications describing (VK) as a proved theorem (see [5, p. 248] and [2, p. 320]). When (4) rather than (VK) is mentioned, it is usually attributed to Vinogradov (see [3, p. 467-468]).

3. Back to the Euler Function: Saltykov’s Mistake

By an amalgamative process (and the facts that his paper is in Russian and difficult to read must also have played a role), Saltykov bears the undeserved reputation of having used an unproved estimate in his proof of (S). I must say that I have never actually seen an assertion to this effect in print, but is has occasionally been mentioned to me, and I think this can be as devastating. In fact, apart from the unproved claim about the zero-free region (VK1), which is at the very end of his paper, Korobov correctly proves all the other results he claims, including his Theorem 1 later used by Saltykov. Incidentally, it is interesting to note that Walfisz, who for his proof of (2) makes use of the same theorem of Korobov, never had this reputation. But Walfisz has the admirable habit–and one very pleasant to his readers–of reproving every single auxiliary result he uses. Such as Korobov's Theorem 1, whose proof can be found in the second paragraph of the second chapter of [19], after having been "*sehr stark überarbeitet*", as Walfisz puts it.

I must admit I first only had a "diagonal" look at Saltykov's paper, discovered in the literature the facts, which I have just set out, and became quite convinced that Saltykov's proof of (S) was correct. What most efficiently carried my conviction is a paper [1] addressing the same historical question, which briefly describes the two related controversies, and in which three co-authors firmly assert that Saltykov's estimate (S) "*is undisputed and is the best to date*".

There is indeed an error in Saltykov's proof of (S); but not where it is generally reputed to be.

The theorem of Korobov's [8], which Saltykov uses applies Vinogradov's method [17] to obtain the estimate

$$\left|\sum_{y=1}^{P} e(f(y))\right| \leq cP^{1-\frac{c_0}{n^2}}, \tag{5}$$

for the Weyl's sum $\sum e(f(y))$, where f is a polynomial $f(y) = \alpha_1 y + \cdots + \alpha_{n+1} y^{n+1}$, in which a subset of $2s-1$ consecutive coefficients, beginning with α_{s+2}, satisfy certain conditions rather long to describe (in particular they are rational numbers), where $\delta n \leq s \leq (n+1)/3$ for some $\delta \in (0, 1/3)$, and where c and c_0 are constants (depending on these conditions and in particular on δ). It should be noted that [8] is the last of a series of 4 articles by Korobov, all published in 1958, and in every one of which, under exactly the same hypotheses, he successively improves his estimate on the Weyl sum in (5). Saltykov's assertion that "*estimate* (5) *is certainly not optimal and could be improved*" is thus perfectly understandable. Prompted by this firm belief he chooses to work under the assumption that, provided the conditions in Korobov's theorem are verified, the estimate

$$\left|\sum_{y=1}^{P} e(f(y))\right| \le \exp(c_1 n^{\gamma_1}) P^{1-\frac{c_0}{n^{\gamma_2}}}, \tag{6}$$

holds for some real numbers $\gamma_1 \ge 0$ and $\gamma_2 \ge 1$ with $1 + \gamma_2 > \gamma_1$. His Theorem 2 states that if (6) is satisfied, then

$$H(x) = O((\log x)^{\gamma}(\log\log x)^{1+\epsilon}), \quad \text{where} \quad \gamma := \frac{\gamma_1 + \gamma_2}{\gamma_1 + \gamma_2 + 1}.$$

But in fact, as of today, we know essentially nothing better than what was known in 1958 with (5), so that the most efficient application of Saltykov's Theorem 2, we can hope for is with $\gamma_1 = 0$ and $\gamma_2 = 2$, that is, with $\gamma = 2/3$: this would yield (S), and this is what Saltykov claims [13, (9)]. But the proof of the essential estimate of his Lemma 2.6 is not correct when $\gamma_1 = 0$: see [10, Section 5] for more details. At best, Saltykov's proof of his Theorem 2 is valid for $\gamma_2 = 2$ and γ_1 an arbitrarily small positive number. In other words,

$$H(x) = O((\log x)^{\frac{2}{3}+\epsilon}) \tag{$1\frac{1}{2}$}$$

is proved, which is better than Walfisz' estimate (1), but not as good as his (2).

In fact, $(1\frac{1}{2})$ is precisely the estimate which Saltykov claims to improve on, and about which he asserts that "*Korobov has indicated the possibility of obtaining it*". Incidently, in order to ensure the validity of Saltykov's proof, it is sufficient (sufficient but probably also necessary) to slightly weaken, in the case where $\gamma_1 = 0$, the hypotheses in his Lemma 2.6 (also see [10, Section 5] for more details). By doing this, Saltykov could have established with his method that

$$H(x) = O((\log x)^{\gamma}(\log\log x)^{1+\epsilon(1-\delta)+\frac{\delta}{\gamma_2+1}}),$$

where $\delta = 1$, if $\gamma_1 = 0$ and $\delta = 0$, otherwise, which for $\gamma_1 = 0$ and $\gamma_2 = 2$ is Walfisz estimate (2).

Hence, it seems that Saltykov's method will not yield a result better than (2) as long as an estimate of type (6) for a value of γ smaller than $2/3$ is not available. This is not at all surprising to me: Indeed, the crucial ingredient in both arguments (Walfisz' and Saltykov's) is the same theorem of Korobov and, in view of Walfisz' constant thoroughness in his work, it seems extremely unlikely to me that he could have failed to fully exploit an auxiliary result.

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